Math 656 • March 26, 2014

Midterm Examination

1) **(16pts)** Find all values of z in polar or Cartesian form, and plot them as points in the complex plane:

(a) $$z=(1+i)^i=e^{i\log(1+i)}=e^{i\left[\ln\sqrt{2}+i\frac{\pi}{4}+2i\pi k\right]}=\boxed{e^{i\overbrace{\frac{\ln 2}{2}}^{\arg z}}\underbrace{e^{-\pi\left(2k+\frac{1}{4}\right)}}_{|z|=a\,r^k}}$$

Points lie on a ray of angle ln2/2, with distance between points forming a geometric sequence

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(b) $$z=\tanh^{-1}(-2)\quad\Rightarrow\quad\tanh z=\frac{e^z-e^{-z}}{e^z+e^{-z}}=-2\quad\Rightarrow\quad e^z-e^{-z}=-2\left(e^z+e^{-z}\right)\ \Big\|\times e^z$$

$$\Rightarrow\ e^{2z}-1=-2\left(e^{2z}+1\right)\quad\Rightarrow\ 3e^{2z}=-1\quad\Rightarrow\ e^{2z}=-\frac{1}{3}$$

$$\Rightarrow\ 2z=\log\left(-\frac{1}{3}\right)=\ln\frac{1}{3}+i\pi+i\pi 2k=-\ln 3+i\pi(1+2k)\quad\Rightarrow\ \boxed{z=-\frac{\ln 3}{2}+i\pi\left(\frac{1}{2}+k\right)}$$

These points lie on a vertical line *x* = –ln3 / 2, with a vertical spacing of *i π*

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2) **(18pts)** For each integral below, describe *all* singularities of the integrand, and use the most convenient method to calculate the integral. If the integral is zero, explain why.

(a) $$\oint_{|z|=1}\frac{dz}{\cosh z}=\boxed{0}\ \text{since zeros of }\cosh z\text{ are located at }z=\underbrace{i\pi\left(\frac{1}{2}+k\right)}_{|z|\geq\pi/2>1},\ k=0,\pm1,\pm2\ldots$$

(b) $$\oint_{|z|=2}\frac{\cos(\pi z)\,dz}{z^2+z}=\oint_{|z|=\varepsilon}\underbrace{\frac{\cos(\pi z)}{z+1}}_{f(z)}\frac{dz}{z}+\oint_{|z+1|=\varepsilon}\underbrace{\frac{\cos(\pi z)}{z}}_{f(z)}\frac{dz}{z+1}=2\pi i\left\{\frac{\cos(0)}{1}+\frac{\cos(-\pi)}{-1}\right\}=\boxed{4\pi i}$$

(c) $$\oint_{|z|=1}\frac{e^z\,dz}{z^3+5iz^2}=\oint_{|z|=2}\underbrace{\frac{e^z}{z+5i}}_{f(z)}\frac{dz}{z^2}=2\pi i\left.\frac{d}{dz}\frac{e^z}{z+5i}\right|_{z=0}=2\pi i\left.\frac{e^z(z+5i-1)}{(z+5i)^2}\right|_{z=0}=2\pi i\,\frac{5i-1}{-25}=\boxed{\frac{2\pi}{25}(i+5)}$$

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3) (14pts) Use the most convenient method to calculate the following integrals over a quarter-circle *C* centered at the origin and connecting point *z*=*i* to *z*=1. Is the integral in part (a) single-valued?

(a) Along C the antiderivative $\frac{1}{3}\Big(\underbrace{z^2-3}_{u}\Big)^{3/2}$ does not intersect the branch cut if we choose $-\pi\le\arg u<\pi$,

so the antiderivative is continuous and we can use the Fundamental Theorem of Calculus

$$\int_C z\Big(\underbrace{z^2-3}_{u}\Big)^{1/2}dz=\frac{1}{2}\int_C u^{1/2}du=\frac{1}{3}u^{3/2}\Big|_{u(i)}^{u(1)}=\frac{1}{3}\left[(-2)^{3/2}-(-4)^{3/2}\right]$$

$$=\frac{\sqrt{2^3}-\sqrt{4^3}}{3}(-1)^{3/2}=\pm e^{i3\pi/2}\,\frac{2\sqrt{2}-8}{3}=\boxed{\pm i\,\frac{2\sqrt{2}-8}{3}}$$

Not single valued - the integrand is double-valued, and so is the result

(b) $$\int_C(\bar{z}+z)\,dz=\int_{\pi/2}^{0}2x\,de^{i\theta}=2\int_{\pi/2}^{0}x\,ie^{i\theta}d\theta=2i\int_{\pi/2}^{0}\cos\theta(\cos\theta+i\sin\theta)\,d\theta=2i\int_{\pi/2}^{0}\cos^2\theta\,d\theta-2\int_{\pi/2}^{0}\cos\theta\sin\theta\,d\theta$$

$$=2i\int_{\pi/2}^{0}\frac{1+\cos2\theta}{2}d\theta-\underbrace{\int_{\pi/2}^{0}\sin2\theta\,d\theta}_{-\frac{1}{2}\cos2\theta\big|_{\pi/2}^{0}=-1}=i\int_{\pi/2}^{0}d\theta+i\underbrace{\int_{\pi/2}^{0}\cos2\theta\,d\theta}_{0}+1=\boxed{1-\frac{i\pi}{2}}$$

4) (13pts) Find the bound on $\left|\int_C \frac{\sin z\,dz}{z+i}\right|$, where the integration contour *C* is a semi-circle of radius 2 connecting points *–2* and *2* in the upper half-plane.

Upper bound on the numerator: $|\sin z| = |\sin x \cosh y + i \cos x \sinh y|$

$$= \sqrt{\sin^2 x\ \cosh^2 y + \cos^2 x\ \sinh^2 y} < \sqrt{\cosh^2 y + \sinh^2 y} = \sqrt{2\sinh^2 y + 1} \le \sqrt{2\sinh^2 2 + 1}$$

Lower bound on denominator: $|z+i| \ge \big||z| - |i|\big| = 2 - 1 = 1$

Length of contour: $\pi \mathrm{R} = 2\pi \Rightarrow$ $\boxed{\left|\int_C \frac{\sin z\,dz}{z+i}\right| \le 2\pi\sqrt{2\sinh^2 2 + 1}}$ Another, less tight bound: $\boxed{\left|\int_C \frac{\sin z\,dz}{z+i}\right| \le 2\pi e^2}$

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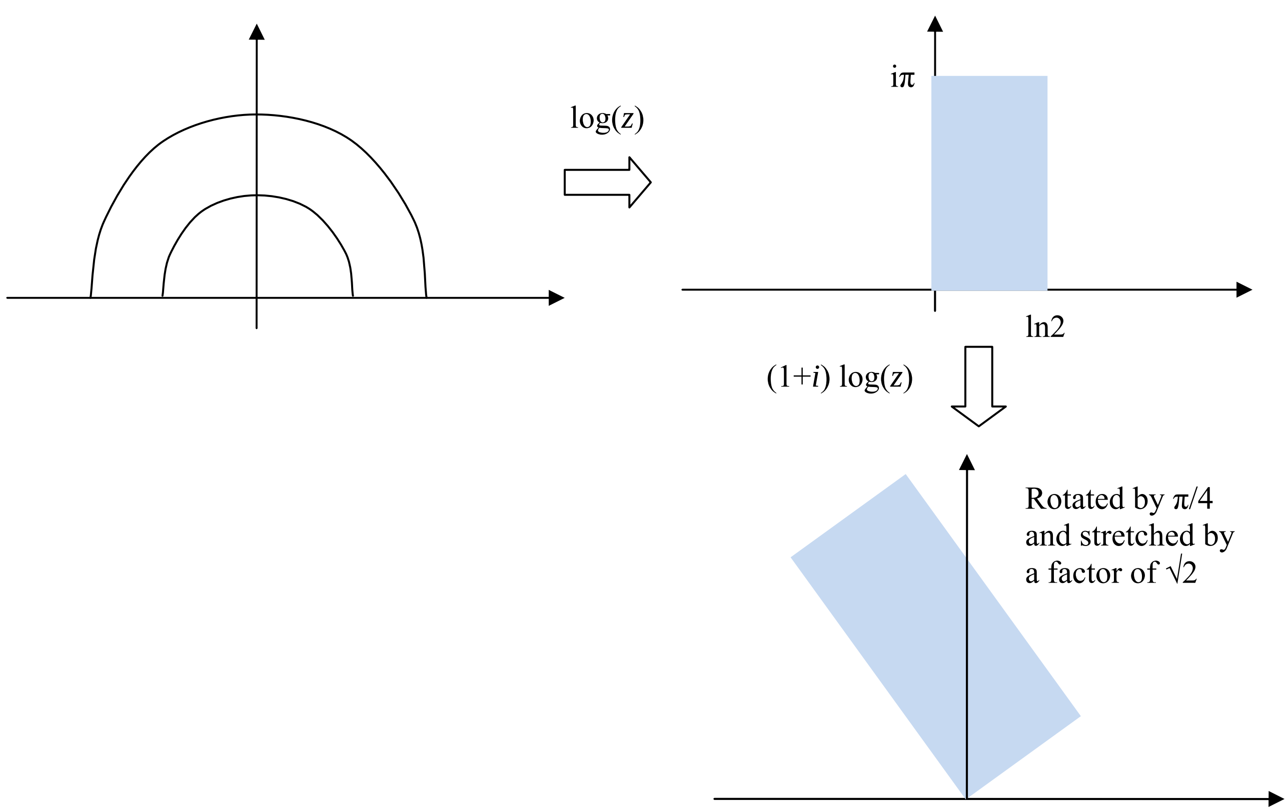
5) (13pts) Find and sketch the image of the half-ring 1<|*z*|<2, 0<arg *z*<π, under the transformation *w* = (*i* +1) log *z*



6) (13pts) Consider the function $f(z) = \mathrm{Re}(z)$. Is this function continuous in any part of complex plane? Is it complex-differentiable anywhere? Examine analyticity directly (using limit definition of derivative), and verify your answer using Cauchy-Riemann equations.

This function is continuous and smooth everywhere but nowhere complex-differentiable:

$$\frac{df}{dz} = \lim_{h\to 0}\frac{\mathrm{Re}(z+h)-\mathrm{Re}(z)}{h} = \begin{cases} \mathrm{Im}(h)=0: & \lim_{h\to 0}\frac{\mathrm{Re}(z)+h-\mathrm{Re}(z)}{h} = 1 \\ \mathrm{Re}(h)=0: & \lim_{h\to 0}\frac{\mathrm{Re}(z)-\mathrm{Re}(z)}{h} = 0 \end{cases} \quad \text{Different limit along different directions} \Rightarrow \text{limit does not exist}$$

Now, using Cauchy-Riemann equations:

$$f(z) = x \quad \Rightarrow \quad \begin{cases} u = x \\ v = 0 \end{cases} \quad \Rightarrow \quad u_x = 1 \neq v_y = 0 \quad \Rightarrow \quad \text{Not analytic anywhere}$$

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7) (13pts) Consider *any* branch of function $\left(z^2+1\right)^{1/3}$, describe its branch cut(s) and describe the discontinuity of this function across the branch cut(s).

$$\left(z^2+1\right)^{1/3} = (z+i)^{1/3}(z-i)^{1/3}$$

Infinitely many choices (will explain in class): pick any two rays outgoing from points $\pm i$

Each of the two factors will introduce a jump factor of $\left(e^{i2\pi}\right)^{1/3} = e^{i2\pi/3} = \frac{-1+i\sqrt{3}}{2}$

These two factors do **not** cancel each other along any common branch cut

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8) (13pts) Suppose that $f(z)$ is entire. Use Cauchy-Riemann identities to prove that function $\mathrm{F}(z) = \overline{f(\overline{z})}$ is also entire

$$\mathrm{F}(z) = \overline{f(\overline{z})} = \overline{u(x,-y)+i\,v(x,-y)} = u(x,-y) - i\,v(x,-y) \quad \Rightarrow \quad \begin{cases} U(x,y) = u(x,-y) \\ V(x,y) = -v(x,-y) \end{cases}$$

$$\begin{cases} U_x = u_x(x,-y); \quad V_y = \frac{d}{dy}\left(-v(x,-y)\right) = v_y(x,-y) \quad \Rightarrow \quad U_x = V_y \ \text{ if } \ u_x = v_y \\ U_y = -u_y(x,-y); \ V_x = -v_x(x,-y) \quad \Rightarrow \quad U_y = -V_x \ \text{ if } \ u_y = -v_x \end{cases}$$